EXACTNESS AND UNIFORM EMBEDDABILITY OF DISCRETE GROUPS

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Abstract. We define a numerical quasi-isometry invariant, $R(\Gamma)$, of a finitely generated group Γ, whose values parametrize the difference between Γ being uniformly embeddable in a Hilbert space and $C_r^*(\Gamma)$ being exact.

1. Introduction

In his study of large scale properties of finitely generated groups, M. Gromov introduced the notion of uniform embeddability. [6]. Recall that a *uniform embedding* of one metric space (X, d_X) into another (Y, d_Y) is a function $f\colon X \to Y$ for which there exist non-decreasing functions $\rho_\pm\colon [0,\infty) \to \mathbb{R}$ such that $\lim_{r\to\infty} \rho_\pm(r) = +\infty$ and such that for all x, $y \in X$

$$\rho_-(d_X(x,y)) \le d_Y(f(x), f(y)) \le \rho_+(d_X(x,y)). \tag{1}$$

The condition $\lim_{r\to\infty} \rho_\pm(r) = +\infty$ is summarized by saying that the $\rho_\pm$ are *proper*. In an appendix we collect several known facts about the relation between uniform embeddings and other notions from coarse geometry.

Gromov raised the question of whether a finitely generated group that is uniformly embeddable in a Hilbert space (when viewed as a metric space with a word length metric) satisfies the Novikov Conjecture [5]. This was answered affirmatively by Yu:

Theorem ([18, 17])**.** *Let Γ be a finitely generated group, equipped with a word length metric. If Γ is uniformly embeddable in Hilbert space then Γ satisfies both the Novikov Conjecture and the Coarse Baum-Connes Conjecture.*

Recently, Gromov has proved the existence of a countable discrete group which is not uniformly embeddable in a Hilbert space [7]. On the other hand, it has been observed that this group does satisfy the Novikov Conjecture, although it is not known whether it satisfies the Coarse Baum-Connes Conjecture [11].

The first author was supported in part by an MSRI Postdoctoral Fellowship and NSF Grant DMS-0071402. The second author was supported in part by NSF Grant DMS-0071435.

From the analytic side, E. Kirchberg and S. Wassermann extensively studied the notion of exactness of a countable discrete group [12, 13]. Recall that Γ is *exact* if $C^*_r(\Gamma)$ is an exact C^*-algebra, that is, if taking minimal tensor product with $C^*_r(\Gamma)$ on each of the terms in a short exact sequence of C^*-algebras preserves the exactness of the sequence.

Uniform embeddability is a geometric property of a group, while exactness is more closely related to harmonic analysis. It is interesting that there is a relation between these notions. Indeed, the connection between these types of properties is related to the Baum-Connes Conjecture. Recently, it was shown that exactness of a countable discrete group implies its uniform embeddability in a Hilbert space.

Theorem ([10, 9, 14])**.** *Let Γ be a finitely generated discrete group. If $C^*_r(\Gamma)$ is an exact C^*-algebra, then Γ, viewed as a metric space with a word length metric, is uniformly embeddable in a Hilbert space.*

One may ask to what extent the converse of this result holds. The question of whether a uniformly embeddable group is exact has been studied from various perspectives (see [2, 8]). In the present paper we introduce a numerical invariant $R(\Gamma)$ of a finitely generated discrete group Γ which can be viewed as parameterizing the difference between the group being exact and being uniformly embeddable in a Hilbert space.

2. The definition of $R(\Gamma)$

Although our primary interest is in uniform embeddings into Hilbert space, we will formulate the basic definitions in the context of general metric spaces. Recall that a function $f : X \to Y$ is *large-scale Lipschitz* if there exist $C > 0$ and $D \geq 0$ such that

$$d_Y(f(x), f(y)) \leq C d_X(x, y) + D. \tag{2}$$

The following example shows that a uniform embedding of a discrete metric space is not necessarily Lipschitz, or even large-scale Lipschitz.

Example 2.1. Let $X = \{\ (n, 1/n), (n, 0) : n = 1, 2, \dots\ \} \subset \mathbb{R}^2$ with the induced metric. Define $f : X \to \mathbb{R}^2$ by $f(n, 1/n) = (n, 1)$ and $f(n, 0) = (n, 0)$. Then f is a both a uniform embedding and a large-scale Lipschitz map but it is not a Lipschitz map. Let $Y = \{\, n^2 : n \in \mathbb{N}\, \} \subset \mathbb{R}$ with the induced metric. Define $g : Y \to \mathbb{R}$ by $g(y) = y^2$. Then g is a uniform embedding but it is not large-scale Lipschitz.

Let $\mathrm{Lip}^{\mathrm{ls}}(X,Y)$ denote the set of large-scale Lipschitz maps from X to Y. Following Gromov [6], define the *compression* ρ_f of $f \in \mathrm{Lip}^{\mathrm{ls}}(X,Y)$ by

$$\rho_f(r) = \inf_{d_X(x,y)\geq r} d_Y(f(x), f(y)) \tag{3}$$

The compression function ρ_f is a non-decreasing, non-negative real-valued function satisfying the first inequality in (1), and has the property that if ρ_- is a another such function then $\rho_- \leq \rho_f$. Consequently, f is a uniform embedding if and only if ρ_f is proper. Always assuming that the metric on X is unbounded, we define a real-valued invariant of X as follows.

Definition 2.2. *Let X be a metric space with an unbounded metric.*

(i) *The asymptotic compression R_f of a large-scale Lipschitz map $f \in \mathrm{Lip}^{\mathrm{ls}}(X,Y)$ is*

$$R_f = \liminf_{r\to\infty} \frac{\log \rho_f^*(r)}{\log r}, \tag{4}$$

where $\rho_f^*(r) = \max\{\, \rho_f(r), 1 \,\}$.

(ii) *The compression of X in Y is*

$$R(X,Y) = \sup\{\, R_f : f \in \mathrm{Lip}^{\mathrm{ls}}(X,Y) \,\}.$$

(iii) *If Y is a Hilbert space, then the Hilbert space compression of X is*

$$R(X) = R(X, \mathcal{H}).$$

Remark. The distinction between ρ_f^* and ρ_f in (4) is not essential, but is meant to eliminate pathology; when ρ_f is unbounded, the definition (4) is unchanged if we replace ρ_f^* by ρ_f. Also, observe that $R_f \geq 0$.

Proposition 2.3. *The compression of X in Y satisfies $R(X,Y) \leq 1$. Indeed, the asymptotic compression of a large-scale Lipschitz map f satisfies $R_f \leq 1$.*

Proof. Let $f \in \mathrm{Lip}^{\mathrm{ls}}(X,Y)$ and let $C > 0$ and $D \geq 0$ be constants as in (2) supplied by fact that f is large-scale Lipschitz. Since X is unbounded, there exist sequences x_n and $y_n \in X$ such that $r_n = d_X(x_n, y_n) \to \infty$. For these r_n we have $\rho_f(r_n) = \inf_{d(x,y)\geq r_n} d_Y(f(x), f(y)) \leq Cr_n + D$ and, for all sufficiently large n, $\rho_f^*(r_n) \leq Cr_n + D$. Hence

$$R_f = \liminf_{r\to\infty} \frac{\log \rho_f^*(r)}{\log r} \leq \liminf_{n\to\infty} \frac{\log(Cr_n + D)}{\log r_n} = 1. \qquad \square$$

Proposition 2.4. *If a metric space X admits an isometric embedding into a metric space Y, then $R(X,Y) = 1$.*

Proof. If $f : X \to Y$ is an isometry, we have $\rho_f(r) \geq r$, hence $R_f = 1$. Thus, $R(X, Y) \geq 1$. By the previous proposition $R(X, Y) = 1$. □

In fact, the same conclusion will follow from the existence of a quasi-isometric embedding of X into a Hilbert space (see Theorem 2.12).

Proposition 2.5. *If a metric space X admits an isometric embedding into the Banach space $l^1(\mathbb{N})$ then $R(X) \geq 1/2$.*

Proof. Let $f : X \to l^1(\mathbb{N})$ be an isometric embedding. Define a function $g : \mathbb{R} \to L^2(\mathbb{R})$ by mapping $x \geq 0$ to the characteristic function of $[0, x]$ and $x < 0$ to the characteristic function of $[x, 0]$. Note that $\|g(x) - g(y)\|^2_{l^2(\mathbb{R})} = |x - y|$. For $\mathbf{x} = (x_1, x_2, \dots) \in l^1(\mathbb{N})$ set $h(\mathbf{x}) = g(x_1) \oplus g(x_2) \oplus \cdots \in L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) \oplus \cdots$. Then it is easily checked that $h \circ f$ is a uniform embedding with $\rho^*_f(r) \geq \sqrt{r}$. Hence, $R(X) \geq 1/2$. □

The same conclusion would follow from the existence of a quasi-isometric embedding of X into $l^1(\mathbb{N})$.

We record here some results which will be proved later in the paper.

Example 2.6. The Hilbert space compression of the metric space obtained from a sequence of expander graphs is zero, as is true of any metric space that is not uniformly embeddable in a Hilbert space (see Proposition 3.1).

Example 2.7. According to Example 2.4 we have $R(\mathbb{Z}) = 1$. Further, $R(\mathbb{Z}^n) = 1$, for all $n \in \mathbb{N}$ (see Proposition 4.1).

Example 2.8. Let $\mathbb{F}_2$ be the free group on two generators. We have $R(\mathbb{F}_2) = 1$ (see Proposition 4.2).

Our primary interest is when our metric space is a finitely generated discrete group Γ, equipped with the left invariant metric induced by the *word length* function associated to a finite, symmetric generating set.

The value $R(\Gamma)$ will be shown to be independent of the particular generating set chosen. Equipped with a metric in this manner, Γ is a geodesic space. In fact it will be important to have a result that holds even for a quasi-geodesic space.

Recall that a discrete metric space X is a *quasi-geodesic space* if there exist $\delta > 0$ and $\lambda \geq 1$ such that for all x and $y \in X$ there exists a sequence $x = x_0, x_1, \dots, x_n = y$ of elements

of X such that

$$\begin{aligned} \sum_1^n d_X(x_{i-1}, x_i) &\le \lambda\, d_X(x, y), \\ d_X(x_{i-1}, x_i) &\le \delta, \quad \text{for all } 1 \le i \le n. \end{aligned} \tag{5}$$

Although a uniform embedding of a discrete metric space is not necessarily large-scale Lipschitz (Example 2.1), a uniform embedding of a quasi-geodesic metric space is.

Proposition 2.9 ([6])**.** *Let X and Y be metric spaces, and assume that X is quasi-geodesic. Let $f : X \to Y$ be a uniform embedding Then f is large-scale Lipschitz.*

Proof. We will only use the existence of ρ_+ (a non-decreasing, non-negative real-valued function satisfying the second inequality in (1)). Let $\lambda \ge 1$ and $\delta > 0$ be the constants supplied by the fact that X is quasi-geodesic. We will show that there exist constants $C > 0$, $D \ge 0$ such that

$$d_Y(f(x), f(y)) \le C d_X(x, y) + D, \quad \text{for all } x,\, y \in X.$$

Let x, $y \in X$ and let $x_0, \ldots, x_n$ be a sequence of elements of X satisfying (5). Extract a subsequence $x_{i_0}, \ldots, x_{i_m}$ as follows: $i_0 = 0$ and, assuming $i_0, \ldots, i_j$ are already defined,

$$i_{j+1} = \begin{cases} \text{the smallest integer } k \text{ such that } d(x_{i_j}, x_k) \ge \delta/2, \\ \text{if such exists; if no such } k \text{ exists put } m = j \text{ and} \\ \text{stop.} \end{cases}$$

The subsequence has the following properties:

(i) $x_{i_0} = x$, $d_X(x_{i_m}, y) \le \delta/2$, and
(ii) $\delta/2 \le d_X(x_{i_{j-1}}, x_{i_j}) \le 3\delta/2$, for $1 \le j \le m$.

We have the following estimates:

$$\begin{aligned} d_Y(f(x), f(y)) &\le \sum_{j=1}^m d_Y(f(x_{i_{j-1}}), f(x_{i_j})) + d_Y(f(x_{i_m}), f(y)) \le m\rho_+(3\delta/2) + \rho_+(\delta/2), \\ m\delta/2 &\le \sum_{j=1}^m d_X(x_{i_{j-1}}, x_{i_j}) \le \sum_{i=1}^n d_X(x_{i-1}, x_i) \le \lambda d_X(x, y). \end{aligned} \tag{6}$$

From the second we conclude that $m \le 2\delta^{-1}\lambda d_X(x, y)$ which, combined with the first, yields

$$d_Y(f(x), f(y)) \le 2\delta^{-1}\lambda\rho_+(3\delta/2)\, d_X(x, y) + \rho_+(\delta/2). \quad \square$$

We next establish the quasi-isometry invariance of the Hilbert space compression. Recall that a function $\varphi\colon X \to Y$ is a *quasi-isometry* if there exist $C > 0$ and $D \geq 0$ such that for all x, $x' \in X$

$$C^{-1}d_X(x, x') - D \leq d_Y(f(x), f(x')) \leq Cd_X(x, x') + D, \tag{7}$$

The spaces X and Y are *quasi-isometric* if there exists a quasi-isometry $\varphi\colon X \to Y$ and $K > 0$ such that φ has K-dense range, meaning that every element of Y is within distance K of an element in the image of φ.

Equivalently, X and Y are quasi-isometric if there exist quasi-isometries $\varphi\colon X \to Y$ and $\psi\colon Y \to X$ and a $K > 0$ such that

$$\begin{aligned} d(\psi\varphi(x), x) &\leq K, \quad \text{for all } x \in X \\ d(\varphi\psi(y), y) &\leq K, \quad \text{for all } y \in Y. \end{aligned}$$

In this case, one calls φ a quasi-isometric equivalence.

Proposition 2.10. *Let $\varphi : X_1 \to X_2$ be a quasi-isometry. Then $R_\varphi = 1$*

Proof. By Proposition 2.3 we have $R_\varphi \leq 1$. If C and D are constants as in (7) supplied by the fact that φ is a quasi-isometry we have $C^{-1}r - D \leq \rho_\varphi(r)$, from which we conclude that $R_\varphi \geq 1$. □

Proposition 2.11. *Let $f \in \mathrm{Lip}^{\mathrm{ls}}(X, Y)$ and $g \in \mathrm{Lip}^{\mathrm{ls}}(Z, X)$. Then $f \circ g \in \mathrm{Lip}^{\mathrm{ls}}(Z, Y)$ and $R_{f\circ g} \geq R_f R_g$.*

Proof. Let f and g be as in the statement. Direct computation shows that $f \circ g$ is large-scale Lipschitz, and further that

$$\rho_{f\circ g}(r) \geq \rho_f(\rho_g(r)),$$

and the same for ρ^*. If the increasing function $\rho_g(r)$ is bounded then $R_g = 0$ and there is nothing to prove. We therefore may assume that $\lim_{r\to\infty} \rho_g(r) = +\infty$. From this and the previous inequality we conclude that

$$\begin{aligned} R_{f\circ g} &= \liminf_{r\to\infty} \left\{ \frac{\log \rho^*_{f\circ g}(r)}{\log r} \right\} \\ &\geq \liminf_{r\to\infty} \left\{ \frac{\log \rho^*_f(\rho_g(r))}{\log(\rho_g(r))} \right\} \left\{ \frac{\log(\rho_g(r))}{\log r} \right\} \\ &\geq R_f R_g. \qquad \square \end{aligned}$$

Theorem 2.12. *Let X_1, X_2 be metric spaces. If there exists a quasi-isometry $\varphi : X_1 \to X_2$ then $R(X_1, Y) \geq R(X_2, Y)$, for every metric space Y.*

Proof. Let $\varphi : X_1 \to X_2$ be a quasi-isometry, and let Y be a metric space. If $f \in \mathrm{Lip}^{\mathrm{ls}}(X_2, Y)$ then $f \circ \varphi \in \mathrm{Lip}^{\mathrm{ls}}(X_1, Y)$ and it follows from Propositions 2.11 and 2.10 that $R_{f\circ\varphi} \geq R_f R_\varphi = R_f$. Thus we have

$$\begin{aligned} R(X_1, Y) &= \sup\{\, R_g : g \in \mathrm{Lip}^{\mathrm{ls}}(X_1, Y) \,\} \geq \sup\{\, R_{f\circ\varphi} : f \in \mathrm{Lip}^{\mathrm{ls}}(X_2, Y) \,\} \\ &\geq \sup\{\, R_f : f \in \mathrm{Lip}^{\mathrm{ls}}(X_2, Y) \,\} = R(X_2, Y). \qquad \square \end{aligned}$$

Corollary 2.13. *If the metric spaces X_1 and X_2 are quasi-isometric then $R(X_1, Y) = R(X_2, Y)$ for every metric space Y.* □

Corollary 2.14. *Let Γ be a finitely generated discrete group. The Hilbert space distortion $R(\Gamma)$ is independent of the finite, symmetric generating set used to define the length function and metric on Γ.*

Proof. Word length metrics associated to finite, symmetric generating sets are quasi-isometric; indeed, the identity provides the required K-dense quasi-isometry [3]. □

3. Uniform embeddings and exactness

In this section we will relate the Hilbert space compression of a metric space X to uniform embeddability, and, in the case of a finitely generated discrete group, to exactness.

Proposition 3.1. *Let X be a metric space. If the Hilbert space compression of X is nonzero then X is uniformly embeddable in Hilbert space.*

Proof. Let X be given with $R(X) > 0$. From the Definition 2.2 of $R(X)$ we see that there exists $\varepsilon > 0$ and a large scale Lipschitz map $f \in \mathrm{Lip}^{\mathrm{ls}}(X, \mathcal{H})$ with asymptotic compression greater than ε:

$$R_f = \liminf_{r\to\infty} \frac{\log \rho_f^*(r)}{\log r} > \varepsilon.$$

In particular, for all sufficiently large r , we have $\log \rho_f^*(r) \geq \frac{\varepsilon}{2} \log r$, hence $\rho_f^*(r) \geq r^{\varepsilon/2}$. Consequently ρ_f^*, and ρ_f, are proper and f is a uniform embedding. □

The main result of this section is the following.

Theorem 3.2. *Let Γ be a finitely generated discrete group. If the Hilbert space compression of Γ is greater than $1/2$ then Γ is exact.*

The proof of the theorem relies on the following characterization of exactness [10, 9, 14]:

Proposition 3.3. *Let Γ be a finitely generated discrete group, equipped with word length and metric associated to a finite, symmetric set of generators. Then Γ is exact if and only if there exists a sequence of positive definite functions, u_n:$\Gamma \times \Gamma \to \mathbb{R}$,*
satisfying

$$\text{for all} \quad C > 0,\ u_n \to 1 \text{ uniformly on the strip } \{\, (s,t) : d(s,t) \leq C \,\} \tag{8}$$

and

$$\text{for all } n, \text{ there exist } R > 0, \text{ such that } u_n(s,t) = 0 \text{ if } d(s,t) \geq R. \qquad \square \tag{9}$$

We refer to (8) as the *convergence condition* and to (9) as the *support condition*; a kernel $\Gamma \times \Gamma \to \mathbb{R}$ satisfying the support condition is of *finite width.*

Under the assumption that $R(\Gamma) > 1/2$ we will construct a sequence of positive definite kernels on $\Gamma \times \Gamma$ satisfying the convergence and support conditions of the proposition.

Given a complex-valued kernel $k : \Gamma \times \Gamma \to \mathbb{C}$, define an operator $\mathrm{Op}(k)$ by convolution:

$$\mathrm{Op}(k)\xi(x) = \sum_{y \in Y} k(x,y)\xi(y), \quad \xi \in l^2(\Gamma). \tag{10}$$

We will need both of the following criteria for the boundedness of $Op(k)$ on $l^2(\Gamma)$, (c.f. [15]).

Proposition 3.4. *Under either of the following conditions $Op(k)$ is a bounded operator.*

(i) *If k is bounded and has finite width then $\mathrm{Op}(k)$ is bounded*
(ii) (*Schur Test*) *Let k be non-negative and real-valued with the property that there exists $C > 0$ such that*

$$\begin{aligned} &\sum_{s \in \Gamma} k(s,t) \leq C, \quad \text{for all } t \in \Gamma \\ &\sum_{t \in \Gamma} k(s,t) \leq C, \quad \text{for all } s \in \Gamma. \end{aligned} \tag{11}$$

Then $\mathrm{Op}(k)$ is bounded and $\|\mathrm{Op}(k)\| \leq C$.

Proof of Theorem 3.2. Let Γ be a finitely generated discrete group equipped with the word length metric associated to a finite symmetric generating set. Assuming that $R(\Gamma) > 1/2$ and arguing as in the proof of Proposition 3.1 conclude that there exists a large-scale Lipschitz map $f \in \mathrm{Lip}^{\mathrm{ls}}(\Gamma, \mathcal{H})$, an $\varepsilon > 0$ and an $r_0 > 0$ such that

$$\rho_f(r) \geq r^{(1+\varepsilon)/2}, \quad \text{for all } r \geq r_0. \tag{12}$$

Define, for $k \geq 1$, a function $u_k : \Gamma \times \Gamma \to \mathbb{R}$ by

$$u_k(s,t) = e^{-\|f(s)-f(t)\|^2 k^{-1}}, \quad \text{for all } s,\, t \in \Gamma.$$

Since the function $\|f(s) - f(t)\|^2$ is of negative type [4], each u_k is positive definite by Schoenberg's theorem [1], and is also *normalized* in the sense that $u_k(s,s) = 1$, for all $s \in \Gamma$. Further, since f is large-scale Lipschitz, the sequence u_k satisfies the convergence condition (8). However, instead of the support condition (9), they possess a weaker decay property. The remainder of the proof will be devoted to approximating the u_k uniformly by *finite width* positive definite kernels so that both the convergence and support conditions hold for the approximants.

Recall that the *uniform Roe algebra*, $C^*_u(\Gamma)$, is the C^*-algebra of bounded operators on $l^2(\Gamma)$ which is the norm closure of the subalgebra of operators generated by $\operatorname{Op}(k)$, where k is a bounded finite width kernel.

Lemma 3.5. *The operators* $\operatorname{Op}(u_k) \in C^*_u(\Gamma)$, *for all* $k \geq 1$.

Proof. We show that for every $\kappa > 0$ the kernel $u : \Gamma \times \Gamma \to \mathbb{C}$ defined by

$$u(s,t) = e^{-\|f(s)-f(t)\|^2 \kappa}, \quad s,\, t \in \Gamma \tag{13}$$

defines an element $\operatorname{Op}(u) \in C^*_u(\Gamma)$. To this end, define, for $n \in \mathbb{N}$

$$k_n(s,t) = \begin{cases} u(s,t), & \text{if } d(s,t) > n \\ 0, & \text{otherwise.} \end{cases}$$

Note that $u - k_n$ is a bounded finite width kernel so that $\operatorname{Op}(u - k_n) \in C^*_u(\Gamma)$. Since $\operatorname{Op}(u) = \operatorname{Op}(u - k_n) + \operatorname{Op}(k_n)$ on compactly supported elements of $l^2(\Gamma)$, it suffices to show that $\|\operatorname{Op}(k_n)\| \to 0$ as $n \to \infty$.

We proceed using the Schur test. Since the k_n are non-negative real-valued and symmetric, it is sufficient to check either one of the inequalities in (11). For this, we will show that there exists a sequence $C_n \to 0$ such that

$$\sum_{t \in \Gamma} k_n(s,t) = \sum_{m > n} \sum_{d(s,t)=m} u(s,t) \leq C_n, \quad \text{for all } s \in \Gamma.$$

This, in turn, follows from the assertion that there exists C such that

$$\sum_{n \geq 0} \sum_{d(s,t)=n} u(s,t) \leq C, \quad \text{for all } s \in \Gamma.$$

To obtain C, let σ be the *spherical growth function* of Γ defined by

$$\sigma(n) = \operatorname{card}\{\, t \in \Gamma : d(t,e) = n \,\}.$$

Denoting by S the fixed generating set of Γ observe that $\sigma(n) \leq (\operatorname{card} S)^n$. Combining (12) and (13) see that if $d(s,t) = n \geq r_0$ then $n^{(1+\varepsilon)/2} \leq \rho_f(n) \leq \|f(s) - f(t)\|$, and also $u(s,t) \leq e^{-\kappa n^{(1+\varepsilon)}}$. Let $m \geq r_0$ be sufficiently large such that $\operatorname{card}(S) < e^{\kappa m^\varepsilon}$. We estimate:

$$\begin{aligned}
\sum_{n\geq 0} \sum_{d(s,t)=n} u(s,t) &= \sum_{n\leq m} \sum_{d(s,t)=n} u(s,t) + \sum_{n>m} \sum_{d(s,t)=n} u(s,t) \\
&\leq \sum_{n\leq m} \sigma(n) + \sum_{n>m} \sum_{d(s,t)=n} e^{-\kappa n^{1+\varepsilon}} \\
&\leq \sum_{n\leq m} \sigma(n) + \sum_{n>m} \sigma(n) e^{-\kappa n^{1+\varepsilon}} \\
&\leq \sum_{n\leq m} \sigma(n) + \sum_{n>m} \left\{ \frac{\operatorname{card}(S)}{e^{\kappa n^\varepsilon}} \right\}^n \\
&\leq \sum_{n\leq m} \sigma(n) + \sum_{n>m} \left\{ \frac{\operatorname{card}(S)}{e^{\kappa m^\varepsilon}} \right\}^n,
\end{aligned} \tag{14}$$

which is both finite and independent of $s \in \Gamma$. We set C equal to the right hand side of the inequality. This completes the proof of the lemma. □

We now complete the proof of the theorem. Since u_k is normalized we have $\|\operatorname{Op}(u_k)\| \geq 1$. It is straightforward to show that since the u_k are positive definite kernels the $\operatorname{Op}(u_k)$ are positive operators. Let $V_k \in C^*_u(\Gamma)$ be the positive square root of $\operatorname{Op}(u_k)$ and let $W_k \in C^*_u(\Gamma)$ be operators represented by finite width kernels and such that $\|V_k - W_k\|\|V_k\| \to 0$. Define kernels $\widehat{u}_k$ by

$$\widehat{u}_k(s,t) = \langle\, W_k\delta_t, W_k\delta_s \,\rangle, \quad s,\, t \in \Gamma.$$

The $\widehat{u}_k$ are positive definite kernels and, since the W_k are represented by finite width kernels, the $\widehat{u}_k$ are themselves finite width kernels. Finally,

$$\begin{aligned}
|u_k(s,t) - \widehat{u}_k(s,t)| &= |\langle\, (\operatorname{Op}(u_k) - W_k^* W_k)\, \delta_t, \delta_s \,\rangle| \\
&\leq \|V_k^* V_k - W_k^* W_k\| \\
&\leq \|V_k - W_k\| \, (\|V_k\| + \|W_k\|) \\
&\leq \|V_k - W_k\| \, (2\|V_k\| + \|V_k - W_k\|)\,,
\end{aligned}$$

which tends to zero as $k \to \infty$. Consequently $u_k - \widehat{u}_k \to 0$ uniformly on $\Gamma \times \Gamma$ and since the u_k satisfy the convergence condition so do the $\widehat{u}_k$. □

This theorem has the following interesting consequence.

Theorem 3.6. *Let $f : \Gamma \to \mathcal{H}$ be a uniform embedding of a finitely generated group into a Hilbert space. Suppose that $f(\Gamma) \subseteq \mathcal{H}$ is a quasi-geodesic space with the induced metric. Then $C_r^*(\Gamma)$ is an exact C^*-algebra.*

Proof. Since f is a uniform embedding Γ is coarsely equivalent to $f(\Gamma)$ by Proposition 6.2. Since both are quasi-geodesic spaces, it follows from Proposition 6.3 that Γ is quasi-isometric to $f(\Gamma)$. But the latter is isometrically embedded in a Hilbert space, so $R(f(\Gamma)) = 1$ by Proposition 2.4. By quasi-isometry invariance, Corollary 2.13, we get $R(\Gamma) = 1$, and hence, by Theorem 3.3, $C_r^*(\Gamma)$ is exact. □

Remark. One might try to argue along the lines of the previous proof to show that any uniformly embeddable group is exact. The first difficulty is that if $f(\Gamma)$ is not quasi-geodesic with the induced metric then there is no way to deduce that $R(\Gamma) = 1$ even though $R(f(\Gamma)) = 1$. However, one need not give up yet. If one could deduce from $R(f(\Gamma)) = 1$ the existence of u_n's satisfying the conditions in Theorem 3.3, then one could pull back the u_n's to $\Gamma \times \Gamma$ and they would also satisfy the necessary condition, thus showing Γ was exact. The difficulty here is that the argument used to construct the u_n's depended on the fact that the spherical growth rate of a group with a word length metric is at most exponential–a fact that need not hold for an arbitrary discrete metric space like $f(\Gamma)$.

4. Behavior of $R(\Gamma)$ under direct sums and certain free products

Let X and Y be metric spaces. Let $X \times Y$ be the cartesian product with the metric

$$d_{X\times Y}\left((x, y), (x', y')\right) = d_X(x, x') + d_Y(y, y').$$

We will obtain a formula for the Hilbert space distortion $R(X \times Y)$ in terms of $R(X)$ and $R(Y)$.

Proposition 4.1. *For metric spaces X and Y we have $R(X \times Y) = \min\{\, R(X), R(Y)\,\}$.*

Proof. Note that, for fixed $y_0 \in Y$, the map $x \longmapsto (x, y_0)$ provides an isometry $X \to X \times Y$. Applying Theorem 2.12 we conclude that $R(X) \geq R(X \times Y)$. Similarly $R(Y) \geq R(X \times Y)$ and so $\min\{\, R(X), R(Y)\,\} \geq R(X \times Y)$.

We must prove the reverse inequality. Assume that $R(X) \leq R(Y)$. Let $\varepsilon > 0$ be given. We will show that there exists a large-scale Lipschitz map $h\colon X \times Y \to \mathcal{H}$ such that $R_h \geq$

$R(X) - \varepsilon$. From this one obtains

$$R(X \times Y) \geq R_h \geq R(X) - \varepsilon = \min\{\, R(X), R(Y)\,\} - \varepsilon,$$

and the desired inequality follows.

According to the definition of $R(X)$ and $R(Y)$ there exist $f \in \mathrm{Lip}^{\mathrm{ls}}(X, \mathcal{H}_X)$ and $g \in \mathrm{Lip}^{\mathrm{ls}}(Y, \mathcal{H}_Y)$ such that

$$\begin{aligned} R_f &\geq R(X) - \varepsilon \\ R_g &\geq R(Y) - \varepsilon \geq R(X) - \varepsilon. \end{aligned}$$

Define $h\colon X \times Y \to \mathcal{H} = \mathcal{H}_X \oplus \mathcal{H}_Y$ by $h(x,y) = f(x) \oplus g(y)$. From the inequality

$$\frac{\alpha + \beta}{\sqrt{2}} \leq \left(\alpha^2 + \beta^2\right)^{1/2} \leq \alpha + \beta, \quad \text{for all } \alpha,\, \beta \geq 0, \tag{15}$$

we conclude that $h \in \mathrm{Lip}^{\mathrm{ls}}(X \times Y, \mathcal{H})$. It remains to estimate the compression of h, again using (15). We have,

$$\begin{aligned} \|h(x,y) - h(x',y')\| &= \|f(x) - f(x') \oplus g(y) - g(y')\| \\ &\geq \frac{1}{\sqrt{2}} \left\{ \|f(x) - f(x')\| + \|g(y) - g(y')\| \right\} \end{aligned}$$

If $d_{X \times Y}((x,y),(x',y')) \geq r$ then at least one of $d_X(x,x')$ or $d_Y(y,y') \geq r/2$. Consequently,

$$\begin{aligned} \rho_h(r) &= \inf\{\, \|h(x,y) - h(x',y')\| : d_{X\times Y}((x,y),(x',y')) \geq r\,\} \\ &\geq \frac{1}{\sqrt{2}} \inf\{\, \|f(x) - f(x')\| + \|g(y) - g(y')\| : d_{X\times Y}((x,y),(x',y')) \geq r\,\} \\ &\geq \frac{1}{\sqrt{2}} \min\left\{\, \rho_f\left(\frac{r}{2}\right), \rho_g\left(\frac{r}{2}\right) \right\} \end{aligned}$$

It follows that,

$$\begin{aligned} R_h &= \liminf_{r\to\infty} \frac{\log \rho_h(r)}{\log r} \\ &\geq \liminf_{r\to\infty} \min\left\{ \frac{\log \rho_f\left(\frac{r}{2}\right)}{\log r}, \frac{\log \rho_g\left(\frac{r}{2}\right)}{\log r} \right\} \\ &= \min\{\, R_f, R_g\,\} \geq R(X) - \varepsilon. \quad \square \end{aligned}$$

We next study a the free product $\mathbb{Z} * \mathbb{Z}$. The calculation of $R(\mathbb{Z} * \mathbb{Z})$ requires a new technique to deform uniform embeddings. It is likely that a variant of this technique will apply to other free products (without amalgam), but we will not not address this in the present paper.

Proposition 4.2. *Let $\mathbb{F}_2$ be the free group on two generators. Then $R(\mathbb{F}_2) = 1$.*

Proof. Let $X = (V, E)$ be the Cayley graph of $\mathbb{F}_2$, $V \cong \mathbb{F}_2$ being the set of vertices and E the set of edges. Let $\mathcal{H} = l^2(E)$. Define

$$f : \mathbb{F}_2 \to \mathcal{H}, \quad f(s) = \delta_{e_1(s)} + \cdots + \delta_{e_k(s)},$$

where δ_e is the Dirac function of the edge e and $e_1(s), \ldots, e_k(s)$ are the edges on the unique path in the Cayley graph from $s \in \mathbb{F}_2$ to the identity $1 \in \mathbb{F}_2$. Note that $k = d(s, 1)$ so that $\|f(s)\| = \sqrt{d(s, 1)}$. Indeed, the following assertions can be verified directly: $\|f(s) - f(t)\| = \sqrt{d(s, t)}$, for all s and $t \in \Gamma$ and $\sqrt{r} \le \rho_f(r) \le \sqrt{r+1}$. Hence, the asymptotic compression of f is $1/2$.

Our strategy for proving the proposition is to produce, by placing appropriate weights into the above formula for f, a family of large-scale Lipschitz embeddings $f_\varepsilon \in \mathrm{Lip}^{\mathrm{ls}}(\mathbb{F}_2, \mathcal{H})$, for $0 < \varepsilon < 1/2$, such that $R_{f_\varepsilon} \to 1$ as $\varepsilon \to 1/2$. Denote $\xi_\varepsilon(x) = x^\varepsilon$ and define weights by $c_{\varepsilon,n} = \xi_\varepsilon(n) = n^\varepsilon$, for $n \in \mathbb{N}$. Define $f_\varepsilon \colon \mathbb{F}_2 \to l^2(E)$ by

$$f_\varepsilon(s) = c_{\varepsilon,1}\delta_{e_1(s)} + \cdots + c_{\varepsilon,k}\delta_{e_k(s)},$$

where k and $e_1(s), \ldots, e_k(s)$ are as above.

In order to show that f_e is a large scale Lipschitz map it suffices to show that there exists $C > 0$ such that

$$d(s, t) = 1 \Longrightarrow \|f_\varepsilon(s) - f_\varepsilon(t)\|^2 \le C, \quad \text{for all } s,\, t \in \mathbb{F}_2.$$

Let s, $t \in \mathbb{F}_2$ be such that $d(s, t) = 1$. Denote by k the length of s and, without loss of generality, $k + 1$ the length of t. We have

$$\|f_\varepsilon(s) - f_\varepsilon(t)\|^2 = c_{\varepsilon,1}^2 + (c_{\varepsilon,2} - c_{\varepsilon,1})^2 + \cdots + (c_{\varepsilon,k+1} - c_{\varepsilon,k})^2$$

so that the desired inequality follows from the elementary fact that $\sum_{j=2}^{\infty}(c_{\varepsilon,j} - c_{\varepsilon,j-1})^2$ is finite. Indeed,

$$\begin{aligned}\sum_{j=2}^{\infty}(c_{\varepsilon,j} - c_{\varepsilon,j-1})^2 &= \sum_{j=2}^{\infty}\left(\int_{j-1}^{j} \xi_\varepsilon'(x)\,dx\right)^2 \le \sum_{j=2}^{\infty}\int_{j-1}^{j} \left(\xi_\varepsilon'(x)\right)^2\,dx \\ &= \int_1^\infty \varepsilon^2 x^{2\varepsilon-2}\,dx = \frac{\varepsilon^2}{1-2\varepsilon}.\end{aligned}$$

To conclude the proof we must show that $R_{f_\varepsilon} \ge 1/2 + \varepsilon$. In view of Definition 2.2 of the asymptotic compression it suffices to show that there exists a constant $C_\varepsilon > 0$, depending

only on ε, such that

$$\|f_\varepsilon(s) - f_\varepsilon(t)\|^2 \geq C_\varepsilon r^{1+2\varepsilon}, \quad \text{for all } s,\, t \in \mathbb{F}_2 \text{ with } d(s,t) \geq r.$$

Indeed, it follows from this that $\rho_{f_\varepsilon}(r) \geq \sqrt{C_\varepsilon} r^{1/2+\varepsilon}$ for all $r \geq 1$ and hence that $R_{f_\varepsilon} = \liminf_{r\to\infty} \frac{\log \rho_{f_\varepsilon}(r)}{\log r} \geq 1/2 + \varepsilon$. To prove the inequality let $s,\, t \in \mathbb{F}_2$ be such that $d(s,t) \geq r$ and assume, without loss of generality, that $d(1,s) \leq d(1,t)$. Denoting the smallest integer greater than $r/2$ by $\#(r/2)$, one checks easily that the edges $e_1(t), \ldots, e_{\#(r/2)}(t)$ appear in the expression for $f_\varepsilon(t)$, but do not appear in that of $f_\varepsilon(s)$. In particular,

$$\|f_\varepsilon(s) - f_\varepsilon(t)\|^2 \geq c_{\varepsilon,1}^2 + \cdots + c_{\varepsilon,\#(r/2)}^2 \geq \int_0^{r/2} \xi_\varepsilon^2(x)\, dx = \frac{r^{2\varepsilon+1}}{(2^{2\varepsilon+1})(2\varepsilon+1)}. \quad \square$$

5. The equivariant case

Incorporating an action of the group Γ the ideas of the previous section yield results about amenability and a-T-menability. To this end, we adapt the previous definitions and results to the equivariant case. Let Γ be a finitely generated discrete group, equipped as usual with a word length metric. Let X be a metric space on which Γ acts by isometries. We define the equivariant Hilbert space compression of X by restricting our attention to Γ-equivariant large-scale Lipschitz maps of X into Hilbert spaces equipped with actions of Γ by *affine isometries*. Precisely, define

$$\mathrm{Lip}^{\mathrm{ls}}_\Gamma(X,\mathcal{H}) = \begin{cases} \Gamma\text{-equivariant large-scale Lipschitz maps} \\ f : X \to \mathcal{H},\ \mathcal{H} \text{ a } \Gamma\text{-Hilbert space;} \end{cases} \tag{16}$$

the definition of the compression and asymptotic compression of $f \in \mathrm{Lip}^{\mathrm{ls}}_\Gamma(X,\mathcal{H})$ are the same as in the non-equivariant case (see (3) and (4), respectively); the Γ-equivariant Hilbert space compression of X is defined by

$$R_\Gamma(X) = \sup\{\, R_f : f \in \mathrm{Lip}^{\mathrm{ls}}_\Gamma(X,\mathcal{H})\,\}.$$

(Compare to Definition 2.2.) With these definitions in place the following analogs of Theorem 2.12 and its corollaries are proved in the same manner.

Theorem 5.1. *Let X and Y be metric spaces on which the countable discrete group Γ acts by isometries. If there exists an equivariant quasi-isometry $X \to Y$ then $R_\Gamma(X) \geq R_\Gamma(Y)$.* $\square$

Corollary 5.2. *Let Γ be a finitely generated discrete group. The invariant $R_\Gamma(\Gamma)$ is independent of the finite symmetric generating set used to define the length function and metric on Γ.* $\square$

Recall that an affine isometric action of Γ on a Hilbert space $\mathcal{H}$ consists of an orthogonal representation $t \longmapsto \pi_t$ of Γ on $\mathcal{H}$ and a function $b : \Gamma \to \mathcal{H}$ satisfying the *cocycle identity*

$$b(st) = \pi_s(b(t)) + b(s); \tag{17}$$

this identity insures that

$$s \mapsto s\cdot \ : \Gamma \to \mathrm{Isom}(\mathcal{H}), \quad s \cdot x = \pi_s(x) + b(s), \quad x \in \mathcal{H}, \tag{18}$$

defines a homomorphism from Γ into the group of affine isometries of $\mathcal{H}$. An affine isometric action is *metrically proper* if for every bounded set $B \subset \mathcal{H}$ the set $\{\, s \in \Gamma : s \cdot B \cap B \neq \emptyset \,\}$ is finite; equivalently, the cocycle b is *proper* in the sense that for every $C > 0$ the set $\{\, s \in \Gamma : \|b(s)\| \leq C \,\}$ is finite. A countable discrete group Γ has the *Haagerup property* if it admits a metrically proper affine isometric action on a Hilbert space. The first part of the next theorem is analogous to Proposition 3.1; the second part is analogous to Theorem 3.2.

Theorem 5.3. *Let Γ be a finitely generated discrete group. If $R_\Gamma(\Gamma) > 0$ then Γ has the Haagerup property. If $R_\Gamma(\Gamma) > \frac{1}{2}$, then Γ is amenable.* □

According to the theorem, if a finitely generated discrete group Γ has an orthogonal representation on a Hilbert space that admits a cocycle b of sufficiently rapid growth then it is amenable. Indeed, suppose that π is an orthogonal action of Γ on $\mathcal{H}$. A cocycle b for π is an element of $\mathrm{Lip}^{\mathrm{ls}}_\Gamma(\Gamma, \mathcal{H})$, where we view Γ as acting on $\mathcal{H}$ by the affine isometric action (18) and on itself by multiplication on the left; the required equivariance follows from the cocycle identity and it is easy to verify that b is large-scale Lipschitz. Further, one has

$$\|b(s) - b(t)\| = \|\pi_t(b(t^{-1}s)\| = \|b(t^{-1}s)\|$$

from which follows that

$$\rho_b(r) = \inf\{\, \|b(s) - b(t)\| : d(s,t) \geq r \,\} = \inf\{\, \|b(s)\| : d(s,e) \geq r \,\}.$$

Thus, if an orthogonal action of Γ on a Hilbert space $\mathcal{H}$ admits a cocycle b for which

$$R_b = \liminf_{r\to\infty} \frac{\log \inf\{\, \|b(s)\| : d(s,e) \geq r \,\}}{\log r} > \frac{1}{2}$$

then it is amenable. In particular, this is the case if the cocycle satisfies $\|b(s)\| \geq (d(s,e))^{1/2+\varepsilon}$ for some $\varepsilon > 0$.

As an illustration consider once again $\Gamma = \mathbb{F}_2$, the free group on two generators. As in the proof of Proposition 4.2, let $X = (V, E)$ be the Cayley graph of $\mathbb{F}_2$, $V \cong \mathbb{F}_2$ being the set of

vertices and E the set of edges. Let $\mathcal{H} = l^2(E)$ be the Hilbert space of real valued functions equipped with an orthogonal action, π, of $\mathbb{F}_2$, and the function

$$b : \mathbb{F}_2 \to l^2(E), \quad b(s) = \begin{cases} \text{characteristic function of the set of} \\ \text{edges on the unique path from } s \text{ to} \\ \text{the identity} \end{cases}$$

satisfies the cocycle identity (17). Consequently, $b \in \mathrm{Lip}^{\mathrm{ls}}_{\Gamma}(\mathbb{F}_2, l^2(E))$, where we equip $l^2(E)$ with the affine isometric action (18).

As remarked earlier,

$$\|b(s) - b(t)\| = \sqrt{d(s,t)},$$

for all s, $t \in \mathbb{F}_2$, and the asymptotic compression of b is $R_b = \liminf_{r\to\infty} \dfrac{\log \rho_b(r)}{\log r} = 1/2$. In particular, the equivariant Hilbert space compression of $\mathbb{F}_2$ satisfies $R_{\mathbb{F}_2}(\mathbb{F}_2) \geq 1/2$. On the other hand, since $\mathbb{F}_2$ is not amenable we have $R_{\mathbb{F}_2}(\mathbb{F}_2) \leq 1/2$. Hence $R_{\mathbb{F}_2}(\mathbb{F}_2) = 1/2$. This should be compared to Proposition 4.2, in which we proved, by deforming the cocycle b, that that $R(\mathbb{F}_2) = 1$.

6. Appendix

In this appendix we review several known relations between uniform embeddings and other notions of coarse geometry [16]. We include some of the elementary proofs for the convenience of the reader.

Let X and Y be metric spaces. A *coarse map* is a function $f : X \to Y$ satisfying the following two conditions:

(i) For every $R > 0$ there exists an $S > 0$ such that

$$d_X(x, x') \leq R \Longrightarrow d_Y(f(x), f(x')) \leq S.$$

(ii) If $B \subseteq Y$ is bounded, then $f^{-1}(B)$ is bounded.

A coarse map $f : X \to Y$ is a *coarse equivalence* if there is a coarse map $g : Y \to X$ and a $K > 0$ such that

$$\begin{aligned} d(gf(x), x) &\leq K, \quad \text{for all } x \in X, \\ d(fg(y), y) &\leq K, \quad \text{for all } y \in Y. \end{aligned} \tag{19}$$

Proposition 6.1. *Let X and Y be metric spaces. A function $f : X \to Y$ is a uniform embedding if and only if it satisfies the following two conditions:*

(i) *For every $R > 0$ there exists an $S > 0$ such that*

$$d_X(x, x') \leq R \Longrightarrow d_Y(f(x), f(x')) \leq S.$$

(ii) *For every $S > 0$ there exists an $R > 0$ such that*

$$d_X(x, x') \geq R \Longrightarrow d_Y(f(x), f(x')) \geq S.$$

Condition (ii) in this proposition implies condition (ii) in the preceding definition so that a uniform embedding is a coarse map. Conversely, coarse map need not be a uniform embedding.

Sketch. Let $f : X \to Y$ be a function satisfying (i) and (ii). By virtue of (i) we may define a non-decreasing, real-valued function ρ_+ by

$$\rho_+(r) = \sup_{d_X(x,y) \leq r} d_Y(f(x), f(y)); \tag{20}$$

we define $\rho_- = \rho_f$ according to (3). These functions satisfy the inequalities on (1). By virtue of (ii), ρ_- is proper, as is ρ_+.

We omit verification of the converse. □

Proposition 6.2. *Let X and Y be metric spaces. A function $f : X \to Y$ is a uniform embedding if and only if it is a coarse equivalence of X with $f(X) \subseteq Y$ with the induced metric.* □

Proposition 6.3. *Let X and Y be quasi-geodesic metric spaces. A function $f : X \to Y$ is a coarse equivalence if and only if it is a quasi-isometric equivalence.*

Proof. A quasi-isometric equivalence is always a coarse equivalence; we show the converse under the assumption that X and Y are quasi-geodesic spaces.

Let $f : X \to Y$ be a coarse equivalence. The function ρ_+ defined in (20) satisfies the second inequality in (1); hence, according to Proposition 2.9, f is large-scale Lipschitz. It remains only to find constants satisfying the first inequality in (7).

Let $g : Y \to X$ be a coarse map satisfying (19) for some $K \geq 0$. Arguing as for f conclude that g is large-scale Lipschitz. Let $C > 0$ and $D \geq 0$ be constants as in the definition (2) of large-scale Lipschitz; that is, satisfying

$$d_X(g(y), g(y')) \leq C d_Y(y, y') + D,$$

for all y, $y' \in Y$. Let x and $x' \in X$ and calculate

$$\begin{aligned} d_X(x, x') &\leq d_X(x, gf(x)) + d_X(gf(x), gf(x')) + d_X(gf(x'), x') \\ &\leq 2K + C d_Y(f(x), f(x')) + D \end{aligned}$$

from which follows that

$$C^{-1}d_X(x,x') - C^{-1}(2K+D) \le d_Y(f(x), f(x')). \qquad \square$$

Proposition 6.4. *Let X and Y be discrete metric spaces with X quasi-geodesic. A uniform embedding $f : X \to Y$ is a quasi-isometry if and only if $f(X)$, with the metric induced from Y, is a quasi-geodesic space.* $\square$

Proof. Let $f : X \to Y$ be a uniform embedding and assume that $f(X)$ is a quasi-geodesic space. By Proposition 6.2, f is a coarse equivalence between X and $f(X)$, and by Proposition 6.3 it is a quasi-isometry.

For the converse, let $f : X \to Y$ be a quasi-isometry and let $C > 0$ and $D \ge 0$ be associated constants. Let $\delta > 0$ and $\lambda \ge 1$ be the constants as in (5) reflecting the fact that X is a quasi-geodesic space. We show that two points in $f(X)$ are connected by a sequence of points satisfying (5) with δ and λ replaced by

$$\delta' = \frac{3\delta C}{2} + D, \qquad \lambda' = \frac{2C(D+\delta')\lambda}{\delta} + 1.$$

Let $f(x)$ and $f(y) \in f(X)$. We may assume that $d_Y(f(x), f(y)) \ge \delta'$, for if not $f(x), f(y)$ is the required sequence. As in the proof of Proposition 2.9 we obtain a sub-sequence of points in X, $x_0, x_1, \ldots, x_m, y$ connecting x and y and satisfying the required estimates. (The subsequence indices are suppressed here for easier reading.) One can show directly that

$$d_Y(f(x_{j-1}), f(x_j)) \le \delta', \quad d_Y(f(x_m), y) \le \delta', \tag{21}$$

so it remains to verify that

$$\sum_{j=1}^{m} d_Y(f(x_{j-1}), f(x_j)) + d_Y(f(x_m), f(y)) \le \lambda' d_Y(f(x), f(y)). \tag{22}$$

From the first inequality in (7) we conclude that $d_X(x,y) \le C(d_Y(f(x), f(y)) + D)$. Since $d_Y(f(x), f(y)) \ge \delta'$ one concludes that

$$d_X(x,y) \le C\left(1 + \frac{D}{\delta'}\right) d_Y(f(x), f(y)).$$

Combining this inequality with (21) and the bound on m from (2.9), the sum in (22) is bounded by

$$(m+1)\delta' \le \left(\frac{2\lambda}{\delta} d_X(x,y) + 1\right)\delta' \le \frac{2\lambda}{\delta} C\left(\delta' + D\right) d_Y(f(x), f(y)) + \delta' \le \lambda' d_Y(f(x), f(y)),$$

where we again use the assumption that $d_Y(f(x), f(y)) \ge \delta'$. This concludes the proof. $\square$

The inclusion of a finitely generated group as a subgroup in another finitely generated group is a uniform embedding, but it's range need not be a quasi-geodesic metric space with the induced metric; the inclusion of $\mathbb{Z}$ in the discrete 3-dimensional Heisenberg group provides an example of this phenomenon.

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